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Multiple trees

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Introduction

In [4], J. Tits and the author introduced the notion of a twin tree. This involved two trees and a function d^*, defined on pairs of vertices, one from one tree, one from the other. An important example arose from the group $\mathrm{GL}_2(A)$ where A is a ring $k[t, t^{-1}]$ of Laurent polynomials. In this paper I shall extend the concept of a twin tree to that of a "multiple tree," involving two or more trees related by a similar function d^*; twin trees are the special case in which there are exactly two trees in the "multiple." These combinatorial objects provide geometries well adapted to the study of $\mathrm{GL}_2(A)$ when A is a ring of rational functions on the projective line having poles at any desired set of rational points (not just at zero and infinity, as is the case for $k[t, t^{-1}]$). They also have potential applications to vector bundles on an algebraic curve, examined in Section 7, following ideas of Serre [7] who in his "*multijumelage*" independently introduced the concept of multiple trees.

Unlike single trees, multiple trees exhibit a certain rigidity reminiscent of spherical buildings, and by analogy one develops a notion of apartments and of root groups. Each apartment contains special subsets called "roots," which lead to "root groups." On the other hand, like single trees, multiple trees have ends "at infinity," and each apartment is spanned by two of these ends (though not all pairs of ends span apartments). Each root belongs to one end or the other. In the case of twin trees, already examined in [4], there are nontrivial commutator relations between root groups for roots belonging to the same end. One of the striking features of multiple trees involving three or more trees is that these commutator relations become trivial. This surprising result is proved in Section 5.

The first section starts with the definition of a multiple tree, but it helps to have in mind the definition of a twin tree from [4]. This is a pair of trees together with a function d^* assigning to each pair of vertices (v, w), one in each tree, a nonnegative integer

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satisfying the following condition. If $d^*(v, w) = n$, then for any vertex w' adjacent to w, $d^*(v, w') = n \pm 1$. Moreover if $n > 0$, then $+1$ occurs for a unique such neighbour of w. The definition of a multiple tree is similar. Given a set of trees, one takes a vertex from each tree and defines d^* on this set of vertices. The properties satisfied by d^* require it to define a twin tree when restricted to two trees from the set. A precise formulation, phrased in terms of a product (or rather a restricted product) of trees, is given in Section 1.

Following this definition, some elementary lemmas are proved showing, in particular, that when there are at least three trees in the multiple then they must be homogeneous and isomorphic to one another. Section 1 goes on to describe the $\mathrm{GL}_2(A)$ example mentioned above.

In Section 2 the notion of ends is introduced. Certain pairs of ends generate apartments, and Theorem 1 proves that the set of apartments uniquely determines the function d^*. Then in Sections 3, 4, and 5 automorphism groups are studied. This starts with a rigidity theorem (Theorem 2) that is a descendent of the rigidity theorem for spherical buildings, given as 4.1.1 in [8]; it generalizes Theorem 4.1 of [4] although the phrasing looks a little different. The notion of roots in a multiple tree is then introduced, and by using the rigidity theorem this leads to the concept of a root group. Each apartment contains two types of roots, corresponding to the two ends of the apartment. When the geometry admits full root groups then, as mentioned above, for roots in the same apartment and having the same end, the root groups commute with one another. This is Theorem 3 and is proved in Section 5.

In the final section of the paper we return to the example of $\mathrm{GL}_2(A)$ but in the broader context of a smooth algebraic curve, rather than just the projective line. The vertices of the "multiple tree" can be represented by vector bundles on the curve, a possibility foreseen by Serre [7]. However, when the genus of the curve is not zero, the function d^* can and does take negative values. The resulting structure is therefore not a multiple tree in the strict sense defined in this paper, and a set of conditions for this more general case is given by Proposition 10.

When J. Tits and the author introduced twin trees in [4] an important aim was to provide a combinatorial framework for Kac–Moody groups of rank 2. The group $\mathrm{GL}_2(k[t, t^{-1}])$ is a Kac–Moody group of affine type, over the field k, and the other affine cases arise similarly from algebraic groups over a ring of Laurent polynomials. Such a ring embeds in a field $k(t)$ and distinguishes two places, zero and infinity, corresponding to the two trees of the twin. Additional places can be used to create multiple trees whose automorphism groups are $\mathrm{GL}_2(A)$ where A is the ring of rational functions having poles only at these places. For rank 2 Kac–Moody groups of hyperbolic type it is natural to ask whether a similar extension is available to a group acting on a multiple tree. However, the work in Section 5 eliminates this possibility for those in which the commutator of two short root groups is a long root group. This suggests that for Kac–Moody groups, multiple trees are a feature of the affine case only.

Of course, one can also consider the possibility of multiple trees without a group action. In [5], J. Tits and the author gave a construction of all twin trees by studying the local structure. The idea was to start with a given tree (necessarily semi-homogeneous) and create a twinning with a second tree by building outwards using local data. To what extent one can start with a homogeneous twin tree and create a triple tree by a similar process I do not know. However, at the end of Section 2 an example is given (with infinitely many trees in the multiple) that cannot be extended to a higher multiple.

1. Definitions and example

Let S be an indexing set, and for each s in S, let T_s be a tree in which each vertex lies on at least two edges. If v is a vertex of the product $\prod T_s$, $s \in S$, then v_s will denote its coordinate in T_s. Two vertices v and w of the product will be called *adjacent*, or more precisely *s-adjacent*, if $v_r = w_r$ for all $r \neq s$ and if v_s is adjacent to w_s in the tree T_s (vertices in a tree are *adjacent* when they are joined by an edge). In this case we refer to vw as an *edge*, or more precisely an *s-edge*. A *path* will mean a sequence of vertices each adjacent to the next, and will be called an *s-path* if all its adjacencies are s-adjacencies.

The distance between two vertices of $\prod T_s$ that are joined by a path is defined as the length of a shortest path joining them. A path between two vertices exists precisely when all but finitely many of their coordinates are equal. In this case the distance between v and w is the sum of the distances between v_s and w_s as s ranges over S. The property of being at finite distance is an equivalence relation on the set of vertices of $\prod T_s$. We now fix, once and for all, one equivalence class, and call it T_S. It will be called a *restricted product* of the trees T_s.

Given such a restricted product T_S and a map d^* from the set of vertices of T_S to the natural numbers $0, 1, 2, 3, \ldots,$ the pair (T_S, d^*) will be called a *multiple tree* if the following two properties are satisfied

(i) if v and w are adjacent vertices then $d^*(w) = d^*(v) \pm 1$; and
(ii) if $d^*(v) > 0$, then for each s in S there is a unique vertex w, s-adjacent to v, with $d^*(w) = d^*(v) + 1$.

When $d^*(v) = n$ we call v an *n-vertex*. The pair (T_S, d^*) will usually be denoted simply by the symbol T.

When S has exactly two elements this is nothing other than a twin tree in the sense of [4]. In fact if $S = \{+, -\}$ then d^* is a codistance function on the pair of trees T_+ and T_-, in the sense of [loc. cit.]. More generally if $\{+, -\}$ is a subset of S then each vertex determines a twinning of T_+ and T_-, as in Proposition 1, for which we need the following notation.

If x is a vertex of T_S, and $r, s \in S$ then we let ${}^r x$, respectively ${}^{rs} x$, denote x without its r-coordinate, respectively its r- and s-coordinates. Thus ${}^r x$ and ${}^{rs} x$ are vertices of $T_{S-\{r\}}$ and $T_{S-\{r,s\}}$, respectively.

1.1. Proposition 1. *Let $\{r, s\}$ be a two-element subset of S and let x be any vertex of T. Then d^* and x define a twinning of T_r and T_s as follows. The codistance between vertices v_r and w_s is given by $d^*(v_r, w_s, {}^{rs} x)$.*

Proof. This is immediate from the definition. □

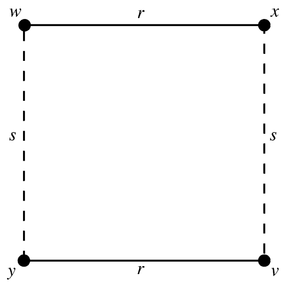

Fig. 1.

1.2. Lemma. *Let x and v be s-adjacent vertices, and let $r \neq s$. Then s-adjacency gives a canonical bijection between the r-neighbourhood of x and the r-neighbourhood of v.*

Proof. This is a simple consequence of the fact that T is a (restricted) direct product. More precisely, writing $x = ({}^{rs}x, x_r, x_s)$, then $v = ({}^{rs}x, x_r, v_s)$ where x_s and v_s are adjacent in T_s. Any r-neighbour u of x has the form $({}^{rs}x, u_r, x_s)$, where u_r and x_r are adjacent in T_r, and u is therefore s-adjacent to a unique r-neighbour of v, namely $({}^{rs}x, u_r, v_s)$. □

Given a vertex w, an r-neighbour x of w, and an s-neighbour y of w, there is a unique vertex v that is s-adjacent to x and r-adjacent to y, as implied by Lemma 1.2—see Fig. 1. The four vertices $\{x, w, y, v\}$ will be called an $\{r, s\}$-*square*, and as notational shorthand we write $v = (w|x, y)$.

In a twin tree (T_+, T_-) two vertices, x_+ in T_+ and x_- in T_-, are called *opposite* if $d^*(x_+, x_-) = 0$. In this case there is a canonical bijection between the neighbourhood of x_+ in T_+ and x_- in T_-, as given in the proof of Proposition 1 in [4]. The following lemma generalizes this result to multiple trees.

1.3. Lemma. *Given a* 0*-vertex z, and distinct elements r and s in S, there is a canonical bijection between the r-neighbourhood of z and the s-neighbourhood of z: each r-neighbour x corresponds to the s-neighbour y for which the fourth vertex on the $\{r, s\}$-square containing x, z and y is a* 2*-vertex.*

Proof. Each r-neighbour x of z is a 1-vertex and has a unique s-neighbour v which is a 2-vertex. Let y be the fourth vertex of the $\{r, s\}$-square containing z, x, and v. Given x, the uniqueness of v implies the uniqueness of y, and sets up a bijection between the r-neighbourhood of z and the s-neighbourhood of z, as required. □

Homogeneity A tree is called *homogeneous* if all its vertices have the same valency, and *semi-homogeneous* if vertices at even distance from one another have the same valency. Before proving the following proposition we define the *s-valency* of a vertex to mean the number of s-edges on that vertex.

1.4. Proposition 2. *When S has cardinality at least* 3 *the trees T_s, for each s in S, are isomorphic to one another and are homogeneous. When S has cardinality* 2 *and the trees are thick, then they are isomorphic and semi-homogeneous.*

Proof. For card $S = 2$ the result was proved in [4, Proposition 1]. Now assume card $S \geqslant 3$, and let $s \in S$. In view of Lemma 1.3 it suffices to show that if z_s and x_s are adjacent vertices of T_s then they have the same valency.

Let z be a vertex of T whose s-coordinate is the chosen vertex z_s in T_s. By altering the other coordinates of z if necessary we may reduce the value of d^* on z, and therefore assume that z is a 0-vertex. Let x be the s-neighbour of z whose s-coordinate is x_s. Let r and t be two distinct elements of S different from s, and let y be a 0-vertex that is t-adjacent to x. By Lemma 1.3 the s-valency of z equals its r-valency, and by Lemma 1.2 this equals the r-valency of x and of y. By Lemma 1.3 again the r-valency of y equals its s-valency, and by Lemma 1.2 again this in turn equals the s-valency of x. This proves that the valency of z_s equals the valency of x_s, and completes the proof. □

Example. $\mathrm{GL}_2(A)$. In [4] the group $\mathrm{GL}_2(k[t, t^{-1}])$, where k is a field, was shown to act on a certain twin tree, and in fact to comprise most of its automorphism group. The ring $k[t, t^{-1}]$—the ring of Laurent polynomials over k—consists of all rational functions on the projective line $\mathbf{P}^1(k)$ having poles only at two points: 0 and ∞. I shall generalize this by taking any non-empty set S of rational points on $\mathbf{P}^1$, using the ring A of rational functions having poles only within the set S, and producing a multiple tree for the group $\mathrm{GL}_2(A)$.

Start with a 2-dimensional vector space over a field K having a discrete valuation v. Let O_v denote the valuation ring of v (elements α of K having $v(\alpha) \geqslant 0$), and let T_v be the tree obtained in the following well-known way (see, e.g., [6]). Each O_v-lattice L in V determines a vertex x_L of the tree T_v, and two lattices L and L' determine the same vertex when they are homothetic (i.e., $L' = \alpha L$ for some element α of K). Let π_v be a uniformizer (generator of O_v), and let $k_v = O_v/\pi_v O_v$ denote the *residue field*. Vertices x_1 and x_2 in T_v are adjacent precisely when there are lattices L_1 for x_1, and L_2 for x_2, with $L_1 \supset L_2 \supset \pi_v L_1$. This implies that the vertices adjacent to x_1 are in a natural bijective correspondence with the 1-spaces in the 2-dimensional k_v-space $L_1/\pi_v L_1$.

For the rest of this section $K = k(t)$, the field of rational functions on the projective line $\mathbf{P}^1(k)$, and V is a 2-dimensional vector space over K. Each rational point of $\mathbf{P}^1(k)$ determines a discrete valuation of $k(t)$ with residue field k, the valuation ring being the local ring at the point. Take two points 0 and ∞, and let T_+ and T_- be the corresponding trees, defined above. In [4] a twinning of T_+ and T_- was given for each $k[t, t^{-1}]$-module M spanning V (note that $k[t, t^{-1}]$ is the ring of rational functions having poles only at 0 or ∞). The subgroup of $\mathrm{GL}_2(V)$ preserving the twinning is $\mathrm{GL}_2(k[t, t^{-1}])$.

Now let S be any set of rational points on $\mathbf{P}^1(k)$, and let A be the ring of rational functions having poles only in S (when S has only two points, A is isomorphic to $k[t, t^{-1}]$). Given a basis ε for the vector space V, the O_s-lattice spanned by ε determines a vertex in T_s, and hence, as s ranges over S, a vertex in the product $\prod T_s$. Let T_S comprise all vertices at finite distance from this vertex. The choice of basis is irrelevant; if ε' is another basis then for all but finitely many s, both ε and ε' span the same O_s-lattice, and hence give vertices of $\prod T_s$ at finite distance. Another way of describing which vertices of the product $\prod T_s$ belong to T_S is to take all S-tuples of O_s-lattices (L_s), $s \in S$, having the property that for all but finitely many s in S the L_s are spanned by a common basis. Of course, if S is finite then T_S equals the product $\prod T_s$.

1.5. Proposition 3. *With the notation above there is a function d^* making T_S a multiple tree such that the group* $\mathrm{GL}(M)$ *acts as an automorphism group of T_S preserving d^*.*

The fact that M determines a function d^* having the required properties will be proved in Section 6 as a corollary to Proposition 10. The definition of d^* is given in the more general context of an algebraic curve, using ideas of Serre [7].

2. Ends and apartments

The purpose of this section is to define the "ends" of a multiple tree T, and define certain subsets of T called "apartments." Every apartment will have two ends, but as in the case of twin trees, not every pair of ends will necessarily give rise to an apartment.

Ends and apartments of a single tree

Given a tree in which every vertex has valency at least 2, an *apartment* is a path without repeated edges or end points (i.e., infinite in both directions). A *half-apartment* is a path with only one end point (and therefore infinite in the other direction). Two half-apartments are said to have the same end if their intersection is a half-apartment. Having the same end is an equivalence relation on the set of half-apartments, and the equivalence classes are called the *ends* of the tree. Each apartment has two ends a and b, and these uniquely determine the apartment, which we denote by (ab). If x is a vertex and a an end, then there is a unique half-apartment having initial vertex x and end a, which we denote by (xa).

Example. Let V be a 2-dimensional vector space over a field K having a discrete valuation with valuation ring O, as in Section 1, and let t be a uniformizing parameter for O. Let X denote the tree for $\mathrm{GL}_2(K)$ using this valuation. Given a basis $\{e_1, e_2\}$ for V, let x_n denote the vertex determined by the O-lattice having basis $\{e_1, t^n e_2\}$. This set of vertices forms an apartment whose ends can be identified with the 1-spaces $\langle e_1\rangle$ and $\langle e_2\rangle$; the half-apartment given by positively increasing n leads to $\langle e_1\rangle$, and the other leads to $\langle e_2\rangle$. In this way two different 1-spaces of V give two different ends of X, but not all ends of X arise in this way unless K is complete with respect to the valuation concerned.

s-paths and ends of T_s

As before T will denote a multiple tree with indexing set S. A path will mean a sequence of vertices, each adjacent to the next. It will be called an *s-path*, for s in S, if each vertex is s-adjacent to the next.

2.1. Proposition 4. *If $\gamma = (x^0, x^1, x^2, \ldots)$ is an s-path such that $d^*(x^0) = n$ and $d^*(x^1) = n - 1$ then d^* decreases monotonically along γ until it reaches x^n (if the path is long enough), at which point it takes the value zero.*

Proof. This follows immediately from the defining property of d^*. $\square$

2.2. Corollary. *If γ is an s-path that is infinite in one direction, then either d^* reaches zero at some vertex, or it increases monotonically along γ. If γ is infinite in both directions, then d^* is zero at some vertex.*

Proof. This is immediate from the previous proposition. $\square$

Given an s-path $\gamma = (\ldots, x^n, x^{n+1}, \ldots)$ the x^i all have the same t-coordinate for any $t \neq s$ in S (i.e., $(x^i)_t = (x^j)_t$ for any two x^i and x^j); call this common coordinate x_t. Writing ${}^s x$, as before, to denote the $(S - \{s\})$-tuple $(x_t)_{t \neq s}$, one has $x^i = ({}^s x, v^i)$, where v^i denotes $(x^i)_s$. The sequence $(\ldots, v^i, v^{i+1}, \ldots)$, denoted γ_s, is a path in T_s, called the *projection* of γ to T_s, and we shall call γ a *lifting* of $(\ldots, v^i, v^{i+1}, \ldots)$ to T_S.

2.3. Lemma. *Let x be any vertex of T and let $(x = a^0, a^1, a^2, \ldots)$ be an r-path, and $(x = b^0, b^1, b^2, \ldots)$ an s-path, along both of which d^* increases monotonically. Write $u^i = (a^i)_r$ and $v^i = (b^i)_s$, so $a^i = ({}^r x, u^i)$ and $b^i = ({}^s x, v^i)$. If $d^*(x) = c$, then $d^*({}^{rs} x, u^i, v^j) = c + i + j$.*

Proof. By Proposition 1, d^* and x determine a twinning of T_r and T_s in which the codistance between two vertices u in T_r and v in T_s is given by $d^*({}^{rs} x, u, v)$. The result is now immediate by [4, (3.3)]. $\square$

2.4. Lemma. *Let x and y be any two vertices of T, let $s \in S$ and let $(v^0, v^1, v^2, \ldots)$ be a half-apartment in T_s. If d^* increases monotonically along the s-path $({}^s x, v^i)$ for $i \geqslant k$, and if n is the distance between ${}^s x$ and ${}^s y$ in $T_{S-\{s\}}$, then d^* increases monotonically along the s-path $({}^s y, v^i)$ for $i \geqslant n + k$.*

Proof. By hypothesis $d^*({}^s x, v^i) > n$ for $i > n + k$. Therefore, $d^*({}^s y, v^i) > 0$ for $i > n + k$. Hence, by Proposition 4, $d^*({}^s y, v^i)$ increases monotonically for $i \geqslant n + k$. $\square$

Definition. Given s in S and a half-apartment $\eta = (v^0, v^1, v^2, \ldots)$ of T_s, the function d^* will be said to *tend to infinity* along η if for some vertex x in T and some integer k, d^* increases monotonically along the s-path $({}^s x, v^i)$ for $i \geqslant k$. By Lemma 2.4 this is independent of the choice of x.

2.5. Lemma. *If two half-apartments of T_s have the same end, then d^* will tend to infinity along both or along neither.*

Proof. Two half-apartments having the same end intersect in a half-apartment, so this is immediate from the definition. $\square$

Ends of multiple trees

In view of Lemma 2.5, the ends of T_s come in two types:

- those ends e_s for which d^* tends to infinity on each half-apartment (x_s, e_s), and
- those ends f_s for which d^* tends to infinity on no half-apartment (x_s, f_s).

(In the second case if $\gamma = (x^0, x^1, x^2, \ldots)$ is any lifting of this half-apartment, then by Proposition 4, d^* takes the value 0 infinitely many times on γ.)

In the first case the set of ends will be denoted $E_s(d^*)$, or simply E_s. If x is a nonzero vertex of T, then by the definition of d^* there is a unique s-path $(x = x^0, x^1, x^2, \ldots)$ along which d^* increases monotonically. This determines an element of E_s denoted $e_s(x)$. Thus x gives a unique end of T_s for each s in S.

2.6. Proposition 5. *Let x and y be nonzero vertices of T. If $e_r(x) = e_r(y)$ for some r in S, then $e_s(x) = e_s(y)$ for all s in S.*

Proof. Suppose $e_r(x) = e_r(y)$, but $e_s(x) \neq e_s(y)$. Since $e_r(x) = e_r(y)$ there is a half-apartment $(u^0, u^1, u^2, \ldots)$ in T_r (leading to $e_r(x)$) such that d^* increases monotonically on both $({}^r x, u^i)$ and $({}^r y, u^i)$ for $i \geqslant 0$.

Now let $A_s = (\ldots, v^2, v^1, v^0 = w^0, w^1, w^2, \ldots)$ be the apartment in T_s from $e_s(x)$ to $e_s(y)$. Then d^* increases monotonically along $({}^s x, v^j)$ and along $({}^s y, w^j)$ for j greater than some suitable bound. By Proposition 1 and (3.3) from [4], this implies that $d^*({}^{rs}x, u^i, v^j) \geqslant i + j - a$, and $d^*({}^{rs}y, u^i, w^j) \geqslant i + j - b$, for all i and j, where a and b are suitable constants. If the distance between x and y is n, then $|d^*({}^{rs}y, u^i, w^j) - d^*({}^{rs}x, u^i, w^j)| \leqslant n$ for all i and j, so $d^*({}^{rs}x, u^i, w^j) \geqslant i + j - c$ for a suitable constant c; for example, $c = b + n$. Now let $k - 1$ be the maximum of a and c, and write $z = ({}^{rs}x, u^k)$, which is a vertex of $T_{S-\{s\}}$. By the inequalities above, we have $d^*(z, v^j) \geqslant j + 1$ and $d^*(z, w^j) \geqslant j + 1$. Therefore, d^* never reaches zero along the s-path (z, A_s), contradicting Corollary 2.2. This contradiction proves that $e_s(x) = e_s(y)$, as required. □

In view of the proposition above each e_r in E_r is associated to a unique e_s in E_s, namely $e_s = e_s(x)$ where x is any vertex for which $e_r = e_r(x)$. We use this to "identify" the sets E_s, for each s in S, with a set E, and note that each nonzero vertex x determines a unique element $e(x)$ in E whose representative in E_s is $e_s(x)$. The elements of E will be called the *ends of the multiple tree*.

2.7. Lemma. *If x and y are adjacent, nonzero vertices then $e(x) = e(y)$.*

Proof. Let s be the type of adjacency between x and y. Without loss of generality $d^*(y) = d^*(x) + 1$ so the unique s-path starting at x along which d^* increases monotonically has y as its second vertex. In particular $e_s(x) = e_s(y)$, completing the proof. □

Remark 1. Given s in S, and any vertex x of T, then by altering the s-coordinate of x one can obtain any end $e(x)$. This is a consequence of Lemma 2.4.

Remark 2. Each vertex x gives a twinning of T_r and T_s, as in Proposition 1. The set of ends of the twinning is naturally identified with E_r and E_s.

Remark 3. The set E_s cannot be the full set of ends of the tree T_s. This follows from the same fact for twin trees, as in [4], using the previous remark. In particular if the valency of each vertex in T_s is finite and at least three, then T_s has uncountably many ends, but the number of vertices of T_s is countable, so E_s is countable by Remark 1.

Remark 4. The tree T_s determines a topology on its set of ends and in this topology E_s is a dense subset (cf. [4, Section 3, Remark 3]). Identifying E_s with E gives an induced topology on E. If $s \neq t$ then the s-topology and the t-topology on E are different. A sequence of distinct ends having a limit point in E_s cannot have a limit in any other set E_t.

Example. In the example of Section 1 the ends are in a natural bijective correspondence with the 1-spaces of the 2-dimensional vector space V—see 2.13 for a more detailed discussion involving apartments, which we now introduce.

Apartments

For each s in S let Σ_s denote a tree in which each vertex has valency two. Such a tree is called "thin," and in the usual terminology of buildings is known as an apartment or a Coxeter complex of type $\tilde{A}_1$. Now let Σ_S denote the restricted product of the Σ_s, for $s \in S$, in the sense meant earlier (i.e., all vertices in the product of the Σ_s at finite distance from some given vertex). An s-path in Σ_S that is infinite in both directions will be called an *s-axis*. If x is any vertex, there is a unique s-axis containing x for each s in S.

If d^* is a function on the vertices of Σ_S, satisfying the conditions given in Section 1, then $\Sigma = (\Sigma_S, d^*)$ is called a *standard S-apartment*. Notice that each vertex lies on a unique s-path that is infinite in both directions, and this uniqueness implies that d^* takes the values $(\ldots, 3, 2, 1, 0, 1, 2, 3, \ldots)$ on such a path. An *apartment* of the multiple tree T will mean an isometric image of Σ in T.

Coordinates. The vertices of a standard apartment will be coordinatized as follows. Let the integers with subscript s $(\ldots(-2)_s, (-1)_s, 0_s, 1_s, 2_s, \ldots)$ be used to denote the vertices of Σ_s. After an appropriate choice of origin 0_s in Σ_s we may assume that $(0_s)_{s\in S}$ is a 0-vertex in Σ. Letting $i(v_s)$ denote the s-coordinate of a vertex v in Σ, then after an appropriate choice of direction (from positive to negative) in Σ_s the function d^* is given by the formula $d^*(v) = |\sum_{s\in S} i(v_s)|$. This can be proved by induction on the number of nonzero coordinates. In particular the 0-vertices are those whose component sum is zero.

The set of zero vertices will be called the 0-*diagonal* of Σ. If S has finite cardinality n then Σ can be regarded as a lattice in n-space, and the points of the 0-diagonal lie in a hyperplane, partitioning the nonzero points of Σ into two sets. This bipartition is true for any cardinality of S.

2.8. Lemma. *The* 0*-diagonal partitions the nonzero vertices of* Σ *into two sets, two vertices being in the same part if and only if they can be joined by a path not containing a* 0*-vertex.*

Proof. Coordinatize the vertices of Σ as above, and for any vertex v in Σ define $i(v) = \sum_{s\in S} i(v_s)$. Let x and y be any two nonzero vertices of Σ. If x_1 is adjacent to x then $i(x_1) = i(x) \pm 1$, so if $i(x)$ and $i(y)$ have different signs, any path from one to the other will pass through a 0-vertex. On the other hand if $i(x)$ and $i(y)$ have the same sign, we produce a path from one to the other not containing a 0-vertex. Let $s_1, \ldots, s_n$ be the coordinates in which x and y differ, and work by induction on n. If $n = 0$ there is nothing to prove. Otherwise let j and k denote the s_n-coordinates for x and y, respectively. Without loss of generality $i(x)$ and $i(y)$ are both positive and $j > k$. Let z be the vertex with the same coordinates as y except that the s_n-coordinate is j instead of k. Then $i(z) = i(y) + j - k$ is positive, and for each point p on the s_n-path from y to z, $i(p)$ lies between $i(y)$ and $i(z)$. Therefore, the s_n-path from y to z contains no 0-vertex. By induction there is also a path in Σ from z to x containing no 0-vertex, and hence a path from x to y having no 0-vertex. $\square$

Two nonzero vertices of Σ will be said to be on the *same side* of the 0-diagonal if there is a path in Σ from one to the other containing no 0-vertices (i.e., not crossing the 0-diagonal). If A is an apartment of T, in other words an isometric image of Σ in T, then an end of A will mean an end $e(x)$ for some nonzero vertex x in A.

2.9. Lemma. *Each apartment of* T *has exactly two ends, and any two vertices on the same side of the* 0*-diagonal have the same end.*

Proof. Two vertices on the same side of the 0-diagonal have the same end by Lemmas 2.7 and 2.8. Moreover, if vertices x and y lie on opposite sides of the 0-diagonal and are adjacent to a common 0-vertex z, then the half-apartment $(z_s, x_s, \ldots)$ leading to $e_s(x)$ is different from $(z_s, y_s, \ldots)$ leading to $e_s(y)$. This shows that $e_s(x) \neq e_s(y)$, so $e(x) \neq e(y)$. $\square$

Quasi-apartments. Any two ends e and f in T determine an apartment $(e_s f_s)$ in the tree T_s. The set of vertices in T whose s-coordinates lie in $(e_s f_s)$, for all s in S, form an isomorphic image of Σ_S in T, that will be denoted (ef), and called a *quasi-apartment*. It is a convex subset of T because if x and y are any two vertices of (ef) then the minimal path between x_s and y_s lies in $(e_s f_s)$, and therefore any minimal path from x to y lies in (ef). It is not necessarily an isometric image of Σ, and therefore not necessarily an apartment of T. A case in which every quasi-apartment is an apartment will be given later in Remark 2.14.

2.10. Proposition 6. *Given a* 0*-vertex* z*, and vertices* x *and* y *adjacent to* z *having distinct ends* $a = e(x)$ *and* $b = e(y)$*, then* (ab) *is an apartment containing* z*,* x*, and* y*.*

Proof. We first show that x, y, and z all lie in the quasi-apartment (ab). In other words x_s, y_s and z_s all lie in $(a_s b_s)$, for any s in S. Given s in S, consider the half-apartments

$\gamma_s = (z_s, a_s)$ and $\delta_s = (z_s, b_s)$ in T_s. Notice that x_s is in γ_s, and y_s is in δ_s, because if x is s-adjacent to z then x_s is the second vertex of γ_s, and if x is not s-adjacent to z then $x_s = z_s$; similarly for y. Now d^* increases monotonically along the s-paths $({}^s z, \gamma_s)$ and $({}^s z, \delta_s)$, and since these s-paths have different ends, the defining property of d^* shows that they have only their initial vertex z in common. Therefore, γ_s and δ_s have no vertex in common except z_s. If $z_s \notin (a_s b_s)$ this would be false because γ_s and δ_s would share further vertices before going their separate ways to the ends a_s and b_s. Therefore, γ_s and δ_s are half-apartments of $(a_s b_s)$, showing that x_s, y_s and z_s all lie in $(a_s b_s)$, as required.

Now label the vertices of $(a_s b_s)$ by the integers, negative to positive going in the direction from a_s to b_s, and with z_s being labelled by 0. For v_s in $(a_s b_s)$ let $i(v_s)$ denote this integer. To show that (ab) is an apartment, as opposed to merely a quasi-apartment, it suffices to prove that

$$d^*(v) = \left|\sum_{s\in S} i(v_s)\right|. \tag{$*$}$$

It is enough to prove this formula whenever the sum $\sum i(v_s) \geqslant 0$, because switching the roles of $+$ and $-$ gives a proof when $\sum i(v_s) \leqslant 0$, and hence a proof for all v.

Step 1. *Proof of* $(*)$ *when* $i(v_s) \geqslant 0$ *for all* s *in* S. Let $s_1, \ldots, s_n$ be those s for which $i(v_s) > 0$. We prove Step 1 by induction on n. If $n = 0$ there is nothing to prove because $v = z$. If $n = 1$ and $s = s_1$ then v lies on the s-path from z in the direction b_s along which d^* increases monotonically. In this case $d^*(v) = i(v_s)$ and $i(v_t) = 0$ for $t \neq s$, proving $(*)$.

Now suppose $n \geqslant 2$, and write $r = s_{n-1}$ and $s = s_n$. Let $x = ({}^r v, 0_r)$, $y = ({}^s v, 0_s)$ and $w = ({}^{rs} v, 0_r, 0_s)$. By induction, $d^*(w)$, $d^*(x)$ and $d^*(y)$ are given by their coordinate sums. Hence d^* increases monotonically along the s-path from w to x, and along the r-path from w to y. If these paths have lengths j and k, respectively, and if $d^*(w) = c$, then we must show that $d^*(v) = c + j + k$. By Proposition 1, d^* and w determine a twinning of T_r and T_s, and so $d^*(v) = c + j + k$ by a result for twin trees, given as (3.3) in [4]. This proves Step 1.

Step 2. *Proof in the general case where* $\sum i(v_s) \geqslant 0$. Let $s_1, \ldots, s_n$ be those s for which $i(v_s) < 0$, and proceed by induction on n. If $n = 0$ this is the case dealt with in Step 1. Now assume Step 2 proven for $m < n$. Let $-k$ be the s_n-coordinate of v, and let w^1 and w^0 be vertices having the same coordinates as v except that the s_n-coordinates are 1 and 0, respectively. By induction $d^*(w^1)$ and $d^*(w^0)$ are given by their coordinate sums, and $d^*(w^0) = d^*(w^1) - 1$. The s_n-path from w^0 to v has length k and therefore by Proposition 4, $d^*(v) = d^*(w^0) - k$. This proves that $d^*(v)$ equals its coordinate sum $\sum i(v_s)$.

2.11. Lemma. *Let z be a* 0*-vertex, and let a, b, and c be three ends such that any two of them span an apartment containing z. Then $(ab) \cap (bc) \cap (ca) = \{z\}$.*

Proof. If the intersection contains any vertex other than z, then by convexity it contains a nonzero vertex x adjacent to z. Since x lies in (ab) one has $e(x) = a$ or b by Lemma 2.9. Similarly for (bc) and (ca), so $e(x) \in \{a, b\} \cap \{b, c\} \cap \{c, a\}$. Therefore, x cannot exist, proving the lemma. □

2.12. Theorem 1. *In a thick multiple tree the set of apartments uniquely determines the function d^*.*

Proof. The definition of d^* implies that $d^*(v) = n$ if and only if v is at distance n from the set of 0-vertices. Therefore, the set of 0-vertices uniquely determines d^*. If z is a 0-vertex, then the thickness assumption ensures there are three distinct vertices x_1, x_2, x_3 that are s-adjacent to z; let $e_i = e(x_i)$. These three ends e_1, e_2, and e_3 are distinct because after projection to T_s they are the ends of three half-apartments starting at z_s, and having distinct second vertices. By Proposition 6, (e_1e_2), (e_2e_3), and (e_3e_1) are apartments. Their intersection contains z, and therefore by Lemma 2.11 they uniquely determine z. Thus the set of apartments determines the set of 0-vertices, completing the proof. □

2.13. Example. In the example of Section 1, the set of apartments is in a natural bijective correspondence with the pairs of 1-spaces $\langle v_1\rangle$, $\langle v_2\rangle$ in V such that $\{v_1, v_2\}$ forms a basis for the module M.

Recall from Proposition 3 that S is a set of rational points on the projective line, and M is a free A-module contained in V, where A is the ring of rational functions having poles only in S. Using $\langle v_1\rangle$ and $\langle v_2\rangle$ the vertices of the apartment can be obtained as follows. As s ranges over S, let $\{i_s\}$ be a set of integers that are nonzero in only finitely many places. Then let $E\{i_s\}$ be the O_s-lattice in V spanned by $\{\pi_s^{i_s} v_1, v_2\}$, where π_s is a uniformizer for O_s. One can show that $d^*(E\{i_s\}) = |\sum i_s|$, so the vertices corresponding to the $E\{i_s\}$ form an apartment. As i_s increases or decreases one obtains s-paths whose ends are $\langle v_2\rangle$ or $\langle v_1\rangle$, respectively. Every apartment is obtained in this way, a fact that can be verified by showing that the group $\mathrm{GL}(M)$ is transitive on the set of apartments (to prove this one can show it is transitive on the set of 0-vertices, and transitive on the set of apartments containing a 0-vertex z).

Before leaving this example note that when k is algebraically closed, every closed point is rational and S can be the whole curve. In this case M is the vector space V, and every pair of 1-spaces forms a basis. In this case *every pair of ends forms an apartment*. This implies that the multiple tree T_S cannot be extended by a product with further trees, because any extension leaves the set of ends constant, but strictly increases the set of apartments. This is now stated as a separate remark.

2.14. Remark. When k is algebraically closed S may be taken as all closed points of the projective line. In this case the example in Section 1 has the property that all pairs of ends form an apartment. In particular the multiple tree cannot be extended by increasing the index set S.

The automorphism group of an apartment

An *automorphism* of a multiple tree X preserves d^* and s-adjacency for each s in S. The automorphism group of an apartment is isomorphic to the automorphism group of the standard apartment Σ, denoted $\operatorname{Aut}\Sigma$.

With the coordinatization above, each 0-vertex v of Σ determines an automorphism by sending x to $x + v$ where $(x + v)_s = x_s + v_s$. The set of 0-vertices form in this way an abelian subgroup of $\operatorname{Aut}\Sigma$ called the *translation subgroup*. It acts freely on the vertices of Σ, and transitively on the set of 0-vertices.

If an automorphism fixes a 0-vertex v of Σ, then for each s in S the direction of the s-axis through v must be either fixed or reversed. If the r-axis is fixed and the s-axis reversed then we obtain a map that fails to preserve d^* (for example, the 0-vertex w having coordinates $w_r = v_r + 1$, $w_s = w_s - 1$, $w_t = v_t$ for $t \neq r, s$ is sent to a 2-vertex). Therefore, either all axes through v are fixed, in which case the automorphism is the identity, or all axes are reversed in which case the automorphism sends x to y where $y_s = 2v_s - x_s$. In particular the stabilizer of a 0-vertex has order 2. We have proved the following.

2.15. Proposition 7. *The group* $\operatorname{Aut}\Sigma$ *is the semi-direct product of its translation subgroup and a group of order* 2 *stabilizing a* 0*-vertex. The translation subgroup acts freely transitively on the set of* 0*-vertices.*

3. A rigidity theorem

The purpose of this section is to prove Theorem 2, showing that certain automorphisms of T are trivial. This theorem is a descendent of the rigidity theorem for spherical buildings, namely Theorem 4.1.1 in [8]. That result was important in the study of root groups, and Theorem 2 will be used in a similar way here. Roots and root groups are defined in the next section. The original result for spherical buildings generalizes to twin buildings (see [9] and [3]), and in particular to twin trees in [4]. The present result contains the twin tree theorem as a special case, though its phrasing looks rather different (cf. 4.1 in [4]).

3.1. Lemma. *Let r and s be distinct elements of S, and let x and y be s-adjacent vertices. Any automorphism fixing x and y and all r-neighbours of x also fixes all r-neighbours of y.*

Proof. This follows from Lemma 1.2. □

3.2. Lemma. *Given a* 0*-vertex z, and s in S, any automorphism fixing z and all its s-neighbours must fix all neighbours of z.*

Proof. This follows from Lemma 1.3. □

Notation. Recall that if w is r-adjacent to x, and s-adjacent to y, then x, w, y are three vertices of an $\{r, s\}$-square whose fourth vertex is denoted $(w \mid x, y)$.

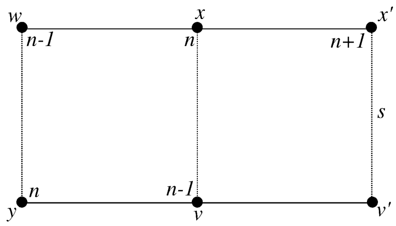

Fig. 2.

3.3. Lemma. *The values of d^* on an $\{r, s\}$-square are, up to a cyclic permutation of the vertices, either $(0, 1, 0, 1)$ or $(n-1, n, n+1, n)$ for some $n \geqslant 1$.*

Proof. Let (w, x, v, y) be the four vertices in a cyclic order (i.e., v and w adjacent to both x and y, as in Fig. 1) with w being r-adjacent to x. Without loss of generality $d^*(w) = n-1$ is minimal among the four vertices. Therefore, $d^*(x) = d^*(y) = n$, and $d^*(v) = n \pm 1$. It suffices to show that if $n \geqslant 2$, then $d^*(v) = n+1$. Suppose on the contrary that $n \geqslant 2$ and $d^*(v) = n-1$.

Now let x' be the unique vertex r-adjacent to x with $d^*(x') = n+1$, and set $v' = (x \mid x', v)$—see Fig. 2. Since v' is adjacent to a vertex of value $n-1$ (namely v) and one of value $n+1$ (namely x'), $d^*(v') = n$. This contradicts Proposition 4 applied to the r-path (y, v, v'). □

3.4. Lemma. *Any two* 0*-vertices are joined by a path consisting only of* 0*- and* 1*-vertices.*

Proof. Let γ be a path between two 0-vertices, and let w be a vertex in γ at which d^* reaches its maximum value m. Let x and y be the vertices of γ preceding and following w, so $d^*(x) = d^*(y) = m-1$. If $m = 1$ there is nothing to prove. If $m > 1$ then it suffices, by an obvious induction, to eliminate the local maximum at w, replacing (x, w, y) by a path between x and y having only vertices of value less than m.

Let r and s be the types of adjacency between w and x, and w and y, respectively. If $r \neq s$ replace (x, w, y) by (x, v, y) where v is the fourth vertex of the $\{r, s\}$-square containing x, w, and y; by Lemma 3.3, $d^*(v) = m-2$. If $r = s$, then let $t \neq s$ and let u be t-adjacent to w with $d^*(u) = m-1$. Let x' and y' be the fourth vertices of the $\{s, t\}$-squares containing w, x, u and w, y, u, respectively; by Lemma 3.3, $d^*(x') = d^*(y') = m-2$. Now replace (x, w, y) by (x, x', u, y', y). □

3.5. Lemma. *If T is a thick multiple tree, then the only automorphism fixing all* 0*-vertices is the identity.*

Proof. Since T_s is a tree, for all s in S, two distinct vertices of T cannot have more than one s-neighbour in common. Therefore, an n-vertex, for $n \geqslant 1$, is uniquely determined by the $(n-1)$-vertices that are s-adjacent to it, there being at least two of them. Induction on n completes the proof. □

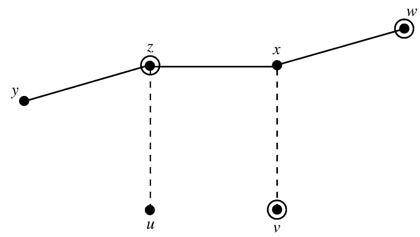

Fig. 3.

3.6. Theorem 2. *Let T be a thick multiple tree, and let z be a* 0*-vertex and zx an s-edge. The only automorphism fixing pointwise the s-neighbourhoods of z and of x is the identity.*

Proof. Let φ be an automorphism as in the statement of the theorem. Given a 0-vertex a and an r-edge ab for some r in S, let $\mathrm{F}(a,b)$ be the hypothesis that all r-neighbourhoods of a and of b are fixed by φ. If this holds for all edges containing a 0-vertex then φ is the identity by Lemma 3.5. The hypothesis of the theorem is $\mathrm{F}(z,x)$, and by Lemma 3.4 and induction along a path containing only 0- and 1-vertices, it suffices to go from the edge zx to any adjacent edge containing a 0-vertex. □

Step 1. *All neighbours of z and x are fixed by φ.* By hypothesis and Lemma 3.2 all neighbours of z are fixed. This, along with Lemma 3.1 applied to the edge zx, implies that all t-neighbours of x are fixed for $t \neq s$; and by hypothesis all s-neighbours of x are fixed. This proves Step 1.

To go from the s-edge zx to an adjacent edge there are four cases to consider: s-edges zy or xw, and t-edges zu or xv, where $t \neq s$ and both v and w are 0-vertices. These are illustrated in Fig. 3.

Step 2. *Proof that* $\mathrm{F}(w,x)$ *holds*. By Step 1 and Lemma 3.1 applied to xw all t-neighbours of w are fixed for $t \neq s$. By Lemma 3.2 therefore all neighbours of w are fixed. This proves $\mathrm{F}(w,x)$.

Step 3. *Proof that* $\mathrm{F}(v,x)$ *holds*. By Lemma 3.1 applied to the edge xv, all s-neighbours of v are fixed. By Lemma 3.2 therefore all neighbours of v are fixed. This proves $\mathrm{F}(v,x)$.

Before going on to the last three steps, let x' and y' be the t-neighbours of z corresponding to the s-neighbours x and y of z under the canonical bijection of Lemma 1.3—i.e., $d^*(z \mid x, x') = d^*(z \mid y, y') = 2$.

Step 4. *Proof that* $\mathrm{F}(z,u)$ *holds for* $u \neq x'$. Since $u \neq x'$, $v = (z \mid x, u)$ is a 0-vertex s-adjacent to u, and t-adjacent to x. By Step 3 all t-neighbours of v are fixed, so Lemma 3.1 applied to the edge uv implies that all t-neighbours of u are fixed. This proves $\mathrm{F}(z,u)$ for $u \neq x'$.

Step 5. *Proof that* $\mathrm{F}(z, y)$ *holds*. By thickness there exists a t-neighbour u of z with $u \neq x', y'$. Then $p = (z \mid y, u)$ is a 0-vertex s-adjacent to u and t-adjacent to y. By Step 4 all t-neighbours of u are fixed, so Lemma 3.1 applied to the edge pu implies that all t-neighbours of p are fixed. By Lemma 3.2 therefore all neighbours of p are fixed. Now Lemma 3.1 applied to the edge py implies that all s-neighbours of y are fixed. This proves $\mathrm{F}(z, y)$.

Step 6. *Proof that* $\mathrm{F}(z, x')$ *holds*. Let $q = (z \mid y, x')$. Since $y \neq x$, q is a 0-vertex s-adjacent to x', and t-adjacent to y. By Step 5 all s-neighbours of y are fixed, so Lemma 3.1 applied to the edge yq implies that all s-neighbours of q are fixed. By Lemma 3.2 therefore all neighbours of q are fixed. Now Lemma 3.1 applied to the edge qx' implies that all t-neighbours of x' are fixed. This proves $\mathrm{F}(z, x')$—the one case missing from Step 4—and completes the proof of the theorem.

3.7. Corollary. *Let X be a quasi-apartment and uv an s-edge in X. The only automorphism fixing pointwise X and the s-neighbours of u and of v is the identity.*

Proof. Let $r \neq s$. By Proposition 2.1 there is an r-path $(v = v^0, \dots, v^n)$ where v^n is a 0-vertex in X. Let $(u = u^0, \dots, u^n)$ be the "parallel" r-path where u^i equals v^i in all coordinates except the s-coordinate (which is u_s for u^i and v_s for v^i); thus v^i and u^i are s-adjacent vertices of X. By Lemma 3.1 applied inductively along the paths $(v = v^0, \dots, v^n = z)$ and $(u = u^0, \dots, u^n = x)$, the automorphism fixes all s-neighbours of v^n and of u^n. The result then follows from Theorem 2 applied to the edge $v^n u^n$. □

4. Roots and root groups

As in the theory of buildings, a "root" will be a subset of an apartment, but not a half-apartment as it usually is. In each apartment (ab) there is a natural partition of its roots, those belonging to the end a, and those belonging to b. In this section and the next, all multiple trees will be thick.

Roots and sectors

Given an apartment (ab) and a vertex v in this apartment the *sector* with *corner* v and *end* b is defined as follows. It comprises all vertices x such that for each s in S, x_s lies in the half-apartment $(v_s b_s)$ of T_s. If $d^*(v) = 0$ the sector is said to have *height* 0, and is called a root. If $d^*(v) = n > 0$, and $e(v) = b$ (respectively a), then the sector with corner v and end b is said to have *height* n (respectively $-n$). Notice that a sector of negative height properly contains a root.

4.1. Lemma. *Let α be a root with corner z and end a. Then every neighbour x of z with $e(x) = a$ lies in α.*

Proof. Suppose x is s-adjacent to z and $e(x) = b$. The s-path from x along which d^* increases monotonically does not contain z, because z is a 0-vertex, so its projection to T_s, namely $(x_s b_s)$, does not contain z_s. Therefore, x_s is the second vertex on the half-apartment $(z_s b_s)$. If $t \neq s$ then $x_t = z_t$, so for any t in S, x_t lies on the half apartment $(z_t b_t)$. Hence $x \in \alpha$ by definition. □

4.2. Proposition 8. *Let α be a root with corner z. Each vertex y adjacent to z and not in α, lies in a unique apartment containing α and y. This gives, for each s in S, a canonical bijection between the set of s-neighbours of z not in α, and the set of apartments containing α.*

Proof. Let b be the end of α, and $a = e(y)$. By Lemma 4.1 $b \neq a$, and by Proposition 6 (ab) is an apartment containing y and z. By definition (ab) comprises all vertices v with v_s in $(a_s b_s)$, for all s; it therefore contains α as a subset since α comprises those v with v_s in $(z_s b_s)$, for all s. Uniqueness is clear since any apartment containing α has b as an end, and any apartment containing y has a as an end.

The second part follows from the first part, and the fact that any apartment containing α contains a unique s-neighbour of z not in α. □

Sector groups

A vertex in a sector θ will be called *s-interior* to θ if it has two s-neighbours in θ. Notice that every vertex of θ, except its corner, is s-interior for some s in S, and a vertex can be s-interior for only finitely many s.

Let U_θ denote the group of automophisms of T fixing θ and every vertex s-adjacent to an s-interior vertex of θ; we shall call it a *sector group*.

4.3. Lemma. *The group U_θ acts freely on the set of apartments containing θ. If θ has negative height then U_θ is the identity.*

Proof. Let A be an apartment containing θ, and let v and w be s-interior vertices of θ that are s-adjacent. An element of U_θ fixes all s-neighbours of v and w. If it also fixes A then it is the identity by Corrolary 3.7. This proves the first statement. If θ has negative height then it properly contains a root, so A is unique by Proposition 4.2. This proves the second statement. □

Root groups

Let α be a root. When the action of U_α is transitive, hence freely transitive, on the set of apartments containing α, we call U_α a *full root group*. If U_α is a full root group for all roots α, then we call the multiple tree *Moufang*. As the next proposition shows, T will be Moufang if U_α is a full root group for every root in a given apartment of T.

4.4. Proposition 9. *Suppose U_α is a full root group for all roots in a given apartment A, and let G be the group generated by the U_α for all α in A. Then G is transitive on the set of all apartments and T is Moufang.*

Proof. Every root β lies in some apartment, so if G is transitive on the set of all apartments then it contains an element g sending β to a root α in A. In that case $U_\beta = g^{-1}U_\alpha g$ is a full root group, so T is Moufang.

To show that G is transitive on the set of all apartments, it suffices by Proposition 6 to take a 0-vertex w and two neighbours u and v, and find an element of G sending (u, w, v) into A.

Step 1. *If x is s-adjacent to a 0-vertex p in A, then there is an element of G fixing p and sending x into A.* Indeed if α is one of the roots in A having corner p then by Proposition 9 there is an element of U_α sending x to the unique s-neighbour of p in A but not in α.

Step 2. *If y is a 0-vertex that is s-adjacent to a vertex y_1 in A, then there is an element of G sending y into A.* Let q be a 0-vertex of A that is t-adjacent to y_1 for some $t \neq s$, and set $q_1 = (y_1 \mid q, y)$, the fourth vertex of the $\{s, t\}$-square containing q, y_1, y. Let α be the root of A having corner q and containing y_1. If $q_1 \in \alpha$ then by convexity $y \in \alpha$, and there is nothing to prove. If $y \in \alpha$ then $q_1 \in \alpha$ and by Step 1 there exists an element g in U_α sending q_1 into A. By convexity $g(y) \in A$.

Step 3. *There is an element of G sending w into A.* If $w \in A$ there is nothing to prove, otherwise let z be a 0-vertex of A and take a path of 0- and 1-vertices from z to w—such a path exists by Lemma 3.4. By Steps 1 and 2, and induction along this path, there is an element of G sending w into A.

Step 4. By Step 3 we may assume $w \in A$. By Step 1 there is an element g in G fixing w and sending u into A. Let α be the root of A with corner w and containing $g(u)$; by Proposition 9 there is an element of U_α sending v into A and fixing w and $g(u)$. This gives an element of G sending (u, w, v) into A, and completes the proof.

4.5. Lemma. *If U_α is a full root group, and v and w are s-interior vertices of α that are s-adjacent, then any automorphism fixing α and all s-neighbours of v and w lies in U_α.*

Proof. Let z be the corner of α and let y be a vertex adjacent to z and not in α. Let g be an automorphism as in the statement of the lemma. By Proposition 9 U_α contains an element h such that $h(y) = g(y)$, and there is a unique apartment containing α and y, which is therefore fixed by $h^{-1}g$. Moreover, $h^{-1}g$ also fixes all s-neighbours of v and w, and therefore by Corollary 3.7 $h^{-1}g$ is the identity. Therefore $g \in U_\alpha$. □

Example. For the projective line in Proposition 3, take an apartment given by 1-spaces $\langle v_1\rangle$ and $\langle v_2\rangle$, as in Example 2.13 where v_1 and v_2 is a basis for the module M. Using v_1 and v_2 as a basis for the matrices in $\mathrm{GL}(M)$, the root groups for this apartment are of the form $\begin{pmatrix} 1 & af \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ af^{-1} & 1 \end{pmatrix}$, where f is a fixed rational function whose divisor is supported on S, and a ranges over k. Assuming an appropriate coordinatization of the apartment, the corner of the root concerned has the divisor of f as its coordinates.

Root reflections. Given an apartment (ab) and a 0-vertex z in this apartment, there is a unique automorphism of (ab) fixing z and switching the two ends a and b—cf. Proposition 7. We call it the *root reflection* centred at z. If α and $-\alpha$ denote the two roots of (ab) having corner z, then given an element u in U_α there exist unique elements v and v' in $U_{-\alpha}$ such that vuv' acts as a root reflection centred at z. This is analogous to the usual theory of apartments and root groups in spherical buildings. In the example above, if u is $\begin{pmatrix} 1 & af \\ 0 & 1 \end{pmatrix}$, then v and v' are both equal to $\begin{pmatrix} 1 & 0 \\ -a^{-1}f^{-1} & 1 \end{pmatrix}$, and the root reflection is represented by the matrix $\begin{pmatrix} 0 & af \\ -a^{-1}f^{-1} & 0 \end{pmatrix}$.

5. Commutator relations between root groups

In this section we intend to prove that if α and β are roots in the same apartment and having the same end, then the root groups U_α and U_β commute with one another. This assumes S has at least three elements, because examples from Kac–Moody groups show it is not necessarily true for twin trees, where S has only two elements.

Standing hypothesis for Section 5. T is Moufang, or in other words if α is a root then U_α is a full root group.

Fix a given apartment (ab) and coordinatize it in the usual way so that the 0-vertices have coordinate sum zero; let increasingly negative coordinates go in the direction of a and increasingly positive ones in the direction of b. We assume without loss of generality that our roots and sectors have end b. To simplify notation I shall make the following two conventions for a vertex v in (ab):

(i) v_s will denote its s-coordinate, an integer previously called $i(v_s)$;
(ii) (v) will refer to the sector having corner v and end b, and $U_{(v)}$ to the corresponding sector group.

Notice that (v) comprises all vertices u such that $u_s \geqslant v_s$ for all s in S.

5.1. Lemma. *The commutator* $[U_{(v)}, U_{(w)}]$ *is a subgroup of* $U_{(x)}$ *where*

$$x_s = \begin{cases} v_s & \text{if } v_s = w_s, \\ \max(v_s, w_s) - 1 & \text{otherwise.} \end{cases}$$

Proof. Let x be as in the statement of the lemma, and let m be defined by $m_s = \max(v_s, w_s)$, so $(v) \cap (w) = (m)$. Let $g \in U_{(v)}$ and $h \in U_{(w)}$, so both g and h fix (m).

Step 1. *If m is s-interior to (x) then $g^{-1}h^{-1}gh$ fixes all s-neighbours of m.* To prove this notice that the s-interiority of m implies that $x_s = m_s - 1$, so without loss of generality that $v_s > w_s$. In this case m is s-interior to the root (w), so h fixes all neighbours of m. Since g fixes m this implies that $g^{-1}h^{-1}gh$ fixes all s-neighbours of m.

Step 2. *$g^{-1}h^{-1}gh$ fixes (x).* For each s in S let mx_s be the vertex having s-coordinate x_s, and other coordinates the same as m. The sector (x) consists of all vertices y for which $y_s \geqslant x_s$, and is therefore the convex hull of the sector (m) and the set of vertices mx_s as s ranges over S. If $mx_s = m$ then it is fixed by g and h. If $mx_s \neq m$ then $x_s = m_s - 1$, so m is an s-interior vertex of (x), hence by Step 1 $g^{-1}h^{-1}gh$ fixes mx_s. This proves Step 2.

Step 3. *If y is s-interior to (x) then $g^{-1}h^{-1}gh$ fixes all s-neighbours of y.* The s-interiority of y implies that $y_s > x_s$, so $y_s \geqslant m_s$. If $y_s > m_s$ then y_s is the s-coordinate of an interior vertex z of (m) and there is a path in (x), not involving s-adjacency, linking z to y. Since $U_{(m)}$ fixes the s-neighbours of z, Lemma 3.1 along with Step 2 shows that it fixes the s-neighbours of y. If $y_s = m_s$, then m is s-interior to (x) and there is a path in (x), not involving s-adjacency, linking m to y. By Lemma 3.1 along with Steps 1 and 2, $g^{-1}h^{-1}gh$ fixes the s-neighbours of y.

Steps 2 and 3 prove the lemma.

5.2. Lemma. *If v and w are distinct 0-vertices such that $v_s \geqslant w_s - 1$ for all s, then $[U_{(v)}, U_{(w)}] = 1$.*

Proof. Let x be as in the statement of Lemma 5.1. If $v_s = w_s$ then $x_s = v_s$; and if $v_s < w_s$ then $v_s = w_s - 1$, so again $x_s = v_s$; finally if $v_s > w_s$ then $x_s = v_s - 1$. Since $v \neq w$ and $\sum v_s = \sum w_s = 0$, one has $v_s > w_s$ for at least one s in S, and therefore $\sum x_s < 0$. By Lemma 4.3 $U_{(x)} = 1$, and the result follows from Lemma 5.1.

5.3. Lemma. *Assume S has at least three elements, and let r, s, t be three distinct elements of S. In the apartment (ab) let α, β, γ be three roots having end b whose corners are, respectively r, s, and t-adjacent to a given 1-vertex v for which $e(v) = b$. Then $U_\alpha \cdot U_\beta \supset U_\gamma$.*

Proof. Let v_α, v_β, v_γ be the corners of α, β, γ, respectively—see Fig. 4.

Notice that v is an s-boundary vertex of both γ and α, and v_β is s-adjacent to v. Therefore, given g in U_γ there exists (a unique) h in U_α such that $g(v_\beta) = h(v_\beta)$. Similarly, starting with h^{-1} in U_α there exists (a unique) f in U_β such that $h^{-1}(v_\gamma) = f(v_\gamma)$. The element hf fixes v_γ and $\alpha \cap \beta$ (a subset of γ), so it fixes their convex hull γ. Moreover, there exist t-adjacent vertices x and y that are t-interior to both α and β, and

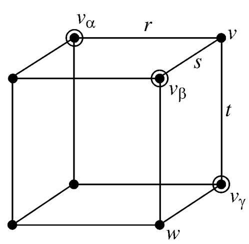

Fig. 4.

hence to γ. Since $h \in U_\alpha$ and $f \in U_\beta$, both h and f fix the t-neighbours of x and y. Therefore, $hf \in U_\gamma$ by Lemma 4.5. Thus g and hf are both elements of U_γ. Moreover, $hf(v_\beta) = h(v_\beta) = g(v_\beta)$, so g and hf have the same effect on the fourth vertex w of the $\{s,t\}$-square containing v, v_γ, and v_β. This vertex w is adjacent to v_γ so Proposition 9 and Lemma 4.3 imply $g = hf$. $\square$

5.4. Theorem 3. *Assume that S has at least three elements. Given two distinct roots α and β in the same apartment and having the same end, the root groups U_α and U_β commute with one another.*

Proof. The proof uses Lemma 5.2 to get started, and then a double induction using Lemma 5.3. Let b denote the common end for α and β, and coordinatize the apartment so that negative to positive coordinates run in the direction of b, and 0-vertices have coordinate sum zero. For a 0-vertex z, the root having corner z and end b will be denoted (z), as in the notation introduced prior to Lemma 5.1.

Let x and y be the corners of α and β, respectively, and let $d_s = y_s - x_s$. Since x and y are 0-vertices, the sum of the d_s is zero, so d_s takes some positive and some negative values; let m be the minimum of these values. If $m = -1$ then the result follows from Lemma 5.2. Assume by induction that the theorem is true for $m = -n$. Now take $m = -(n+1)$.

Step 1. *Suppose that $d_t = -(n+1)$, and $d_s \geqslant -n$* for all other coordinates s.

Step 1A. *Assume there are two coordinates r and s with $d_r, d_s \geqslant -(n-1)$.* Let y' be the same as y except that its t-coordinate is up by 1, and the r-coordinate is down by 1; and let y'' be the same as y except that the t-coordinate is up by 1, and the s-coordinate is down by 1. Then y, y', and y'' are the corners of three roots satisfying the conditions of Lemma 5.3, so $U_{(y')} \cdot U_{(y'')} \supset U_{(y)}$. Moreover, $y'_q - x_q$ and $y''_q - x_q$ are both $\geqslant -n$ for all coordinates q, so by the induction hypothesis both $U_{(y')}$ and $U_{(y'')}$ commute with $U_{(x)}$. Therefore, $U_{(y)}$ commutes with $U_{(x)}$.

Step 1B. *Assume there are not two coordinates q with $d_q \geqslant -(n-1)$.* In this case $d_r > 0$ for one coordinate r, and $d_s = -n$ for all $s \neq t, r$. Now let x' be the same as x except that the r-coordinate is up by 1, and the s-coordinate is down by 1; and let x'' be the same as x except that the r-coordinate is up by 1, and the t-coordinate is down by 1. Then x, x', and x'' are the corners of three roots satisfying the conditions of Lemma 5.3, so $U_{(x')} \cdot U_{(x'')} \supset U_{(x)}$.

The definition of x' implies that $y_q - x'_q = y_q - x_q$ for $q \neq r, s$; $y_r - x'_r = d_r - 1 \geqslant 0$; and $y_s - x'_s = d_s + 1 \geqslant -(n-1)$. Therefore, Step 1A implies that $U_{(x')}$ commutes with $U_{(y)}$. Similarly the definition of x'' implies that $y_q - x''_q = y_q - x_q$ for $q \neq r, t$; $y_r - x''_r = d_r - 1 \geqslant 0$; and $y_t - x''_t = d_t + 1 = -n$. Therefore, $y_q - x''_q \geqslant n$ for all q, so the original induction hypothesis implies that $U_{(x'')}$ commutes with $U_{(y)}$. Since we have shown that $U_{(x')}$ and $U_{(x'')}$ commute with $U_{(y)}$, the previous paragraph shows that $U_{(x)}$ commutes with $U_{(y)}$. This completes the proof of Step 1.

Step 2. Suppose, as in the induction hypothesis, that $m = -(n+1)$, and that the theorem holds whenever $d_s = -(n+1)$ for at most $k-1$ coordinates. Now suppose $d_s = -(n+1)$ for k coordinates. By Step 1 we can assume $k \geqslant 2$, so we may choose r, s, t such that $d_r > 0$, and $d_s = d_t = -(n+1)$. Now let x' be the same as x except that the r-coordinate is up by 1, and the s-coordinate is down by 1; and let x'' be the same as x except that the r-coordinate is up by 1, and the t-coordinate is down by 1. Then $y_q - x'_q \geqslant -(n+1)$ for all q, and $y_q - x'_q = -(n+1)$ for exactly $k-1$ coordinates q; similarly for $y_q - x''_q$. By the Step 2 induction therefore $U_{(x')}$ and $U_{(x'')}$ both commute with $U_{(y)}$. Moreover x, x', and x'' are the corners of three roots satisfying the conditions of Lemma 5.3, so $U_{(x')} \cdot U_{(x'')} \supset U_{(x)}$. Therefore $U_{(x)}$ commutes with $U_{(y)}$.

This completes the second induction, since $d_s \neq 0$ for only finitely many s. Thus U_α and U_β commute.

6. Vector bundles on curves and the function d^*

In this final section I shall return to the example given in Section 1, but in a more general context. Let K be the function field of a smooth projective curve C that is absolutely irreducible over a field k. Each rational point of C determines a discrete valuation of K with residue field k, its valuation ring being the local ring at the point concerned. Let S be any set of rational points on C, and let A denote the ring of rational functions having poles only at points in S. For each s in S let O_s denote the valuation ring at s (i.e., the ring of rational functions having no pole at s), and let T_s denote the corresponding tree. The example given at the end of Section 1 was the special case where C is the projective line.

A vector bundle E on C can be described by giving, for each point s of C, the localization E_s of its sheaf of sections. When s is the generic point, E_s is a vector space V over K, and when s is a closed point E_s is an O_s-lattice in V. The definition of a vector bundle includes a local triviality condition that is equivalent to requiring the lattices E_s be spanned by a common basis for all but finitely many s.

As in [2] the term *adelic lattice* means a set $L = (L_s)$ of O_s-lattices, one for each closed point s of C, such that each lattice spans V and for all but finitely many s the L_s are spanned by a common basis ε. Each vector bundle determines an adelic lattice, and conversely given an adelic lattice L there is a vector bundle E_L whose sheaf of sections, localized at s, is L_s. Let T_C denote the restricted product T_S where S is the set of all closed points on C. The following lemma is the rank 2 case of a more general statement for higher rank vector bundles and adelic buildings (see [2, Section 11]).

6.1. Lemma. *A rank 2 vector bundle E on C determines a vertex of T_C, and two vector bundles E and E' determine the same vertex precisely when $E' = E \otimes F$ for some line bundle F.*

Proof. Let L and L' be the adelic lattices for E and E', respectively. They give the same vertex of T_C when, for each s, one has $L'_s = a_s L_s$ (the a_s being 1 at all but finitely many places). In this case $a = (a_s)$ is a divisor and its corresponding line bundle $F = E_a$ has the property that $F \otimes E = E'$. Conversely if $F = E_a$ is any line bundle, and $E' = F \otimes E$ then $E' = E_{aL}$, so $L' = aL$.

Remark. The orbits of $\mathrm{GL}_2(K)$ on T_C are in bijective correspondence with isomorphism classes of rank 2 vector bundles modulo the Picard group of line bundles.

The function d^*

Let E be a rank 2 vector bundle, and F a rank 1 sub-bundle on a smooth projective curve C. Following Serre [6, p. 100] one defines an integer $N(E; F)$ as follows:

$$N(E; F) = \deg F - \deg E/F = 2\deg F - \deg E.$$

One then defines an integer invariant $d^*(E)$ as follows: $d^*(E) = \sup N(E; F)$, as F ranges over the sub-bundles of rank 1 in E (in [7] $d^*(E)$ is called $N(E)$).

Earlier work of Mumford [1] introduced the notion of *stable* vector bundles. This has been extended to a notion of *semistability*, and a bundle that fails to be semistable is called *unstable*. In terms of the function d^* one has:

$$d^*(E) \begin{cases} < 0 & \text{if } E \text{ is stable,} \\ \leqslant 0 & \text{if } E \text{ is semistable,} \\ > 0 & \text{if } E \text{ is unstable.} \end{cases}$$

Remarks.

(i) If L is a line bundle then $N(E; F) = N(E \otimes L; F \otimes L)$ and hence $d^*(E) = d^*(E \otimes L)$.

(ii) $N(E; F) \equiv \deg E \pmod 2$.

(iii) $d^*(E) \geqslant -2g$, where g is the genus of C—see [6, p. 100].

(iv) If $d^*(E) > 2g - 2$ (which is always the case if $g = 0$), then E decomposes as a direct sum $F \oplus F'$ of line-bundles—see [6, p. 102].

6.2. Lemma. *Let E be a rank 2 vector bundle on the curve C. If E contains sub-bundles L and F such that $N(E; F) > 0$, and $N(E; L) \geqslant 0$, then $L = F$. In particular if E contains a sub-bundle L such that $N(E; L) \geqslant 0$, then $d^*(E) = N(E; L)$.*

Proof. Assume without loss of generality that $N(E; F) \geqslant N(E; L)$, so $\deg L \leqslant \deg F$. Tensoring E, F, and L with F^{-1} allows one to assume that F is generated by a single nonzero section f. Since $\deg F$ is now zero, and $N(E; F) > 0$, one has $\deg E < 0$, and $\deg L \leqslant 0$. If $\deg L < 0$, then the assumption $(E; L) \geqslant 0$ implies that $\deg E/L < 0$; and if $\deg L = 0$ then $N(E; L) = N(E; F) > 0$, so again $\deg E/L < 0$. Either way $H^0(C, E/L) = 0$, showing that the exact sequence

$$0 \to H^0(C, L) \to H^0(C, E) \to H^0(C, E/L)$$

collapses on the right, which implies that f lies in L, so $L \supset F$. Therefore, $L = F$ since both are rank 1 sub-bundles of E. $\square$

Remark. The lemma above implies that if $d^*(E) > 0$, then there is a unique sub-bundle F of rank 1 for which $d^*(E) = N(E; F)$.

As mentioned above, each rank 2 vector bundle E determines a vertex x_E of T_C, and we define $d^*(x_E)$ to be $d^*(E)$. This is well-defined because if E and E' determine the same vertex then $E' = E \otimes L$ for some line bundle L, and $d^*(E \otimes L) = d^*(E)$.

Given a point s of C the vertices of T_C that are s-adjacent to x_E are in a natural bijective correspondence with the 1-spaces D in $E_s/\pi_s E_s$ in the following sense. The 1-space D corresponds to an O_s-lattice D_s such that $E_s \supset D_s \supset \pi_s E_s$, and this defines a vector bundle E_D given by $(E_D)_r = E_r$ for all points $r \neq s$, and $(E_D)_s = D_s$. The vector bundles E and E_D determine points of T_C that are s-adjacent.

Given a rank 1 sub-bundle F of E, then F_D means the sub-bundle of E_D for which $(F_D)_r = F_r$ for all $r \neq s$, and $(F_D)_s = F_s \cap D_s$. If s is a rational point and if $F_s \neq D_s$ then $\deg F_D = \deg F - 1$.

6.3. Lemma. *With the notation above, $N(E_D; F_D)$ equals $N(E; F) + 1$ if $D_s = F_s$, and $N(E; F) - 1$ otherwise.*

Proof. Write $n = N(E; F)$. The definition of E_D implies that $\deg E_D = \deg E - 1$, and by definition $N(E_D; F_D) = 2 \deg F_D - \deg E_D$. There are two cases: if $F_s = D_s$, then $F_D = F$ hence $N(E_D; F_D) = n + 1$; otherwise $\deg F_D = \deg F - 1$, hence $N(E_D; F_D) = n - 1$. $\square$

6.4. Lemma. *Let s be a rational point of C, and let v and v' denote s-adjacent vertices in T_C. If $d^*(v) = n$, then $d^*(v') = n \pm 1$. Moreover, $+1$ occurs for at least one s-neighbour of v, and if $n > 0$ then $+1$ occurs for a unique s-neighbour of v.*

Proof. Without loss of generality v and v' may be represented by vector bundles E and E_D where D is a 1-space D in $E_s/\pi_s E_s$. Let F be a sub-bundle of E such that $N(E; F) = d^*(E) = n$. By Lemma 6.3 $N(E_D; F_D) = n \pm 1$, and therefore $d^*(E_D) \geqslant n - 1$. Reversing the roles of E and E_D implies that $d^*(E_D) \leqslant n + 1$. By Remark (ii) above, $d^*(E_D) \neq n$, proving that $d^*(v') = n \pm 1$. Moreover, by Lemma 6.3 again, $+1$ occurs for at least one v'. Now suppose $n > 0$, so $N(E_D; F_D) \geqslant 0$. Then by Lemma 6.2 $d^*(E_D) = N(E_D; F_D)$, and the result follows from Lemma 6.3. $\square$

S-lattices and T_S

Now let S be a set of rational points on C, and let A be the ring of rational functions having poles only in S. Let V be a given 2-dimensional vector space over K (the field of rational functions on C). The term *S-lattice in V* will mean a collection of O_s-lattices, one for each s in S, such that each lattice spans V and for all but finitely many s the L_s are spanned by a common basis ε. Regarding C–S as $\operatorname{Spec} A$, a vector bundle on C–S is the same thing as a projective A-module. Therefore, a vector bundle on C is nothing other than a projective A-module in V along with an S-lattice in V.

If $L=(L_s)_{s\in S}$ and $L'=(L'_s)_{s\in S}$ are S-lattices that determine the same vertex in T_S then for each s in S, $L'_s=\pi_s^{n_s}L_s$ where $n_s=0$ for all but finitely many s. Let D be the line bundle whose associated divisor is n_s for s in S, and 0 outside S. The S-lattices L and L', along with a given projective A-module M determine vector bundles E and E'. Clearly $E'=E\otimes D$, so $d^*(E)=d^*(E')$. In other words given a projective A-module M in V, and a vertex x in T_S, $d^*(E)$ is an invariant of the pair (M,x). We call it $d^*(M,x)$.

6.5. Proposition 10. *Let S be a set of rational points on a smooth curve C, and let A be the ring of rational functions having poles only in S. Given a projective A-module M, the function $d^*(M,v)$ has the following properties, where v and v' are s-adjacent vertices of T_S.*

(i) *If $d^*(M,v)=n$, then $d^*(M,v')=n\pm 1$.*

(ii) *Moreover, $+1$ occurs for at least one neighbour of v, and if $n>0$ then $+1$ occurs for a unique s-neighbour of v.*

(iii) *$d^*(M,v)\geqslant -2g$ where g is the genus of C.*

Furthermore, the group $\mathrm{GL}(M)$ *preserves the function d^* on T_S.*

Proof. Parts (i) and (ii) are immediate consequences of Lemma 6.4, and part (iii) follows from Remark (iii) above. To prove the statement about $\mathrm{GL}(M)$, let g be any element of $\mathrm{GL}(V)$, and let x be a vertex of T_S. Let L be an S-lattice determining x, and let E be the vector bundle determined by M and L. Since gE is isomorphic to E, one has $d^*(gE)=d^*(E)$, and therefore $d^*(gM,gx)=d^*(M,x)$. If $g\in\mathrm{GL}(M)$ then $gM=M$, completing the proof.

Remark. Proposition 3, which says that (T_S,d^*) is a multiple tree when C is the projective line, is an immediate corollary to Proposition 10.

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